



THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

A characterisation of the least-fixed-point operator by dinaturality

Citation for published version:

Simpson, A 1993, 'A characterisation of the least-fixed-point operator by dinaturality' Theoretical Computer Science, vol 118, no. 2, pp. 301-314., 10.1016/0304-3975(93)90112-7

Digital Object Identifier (DOI):

[10.1016/0304-3975\(93\)90112-7](https://doi.org/10.1016/0304-3975(93)90112-7)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Publisher final version (usually the publisher pdf)

Published In:

Theoretical Computer Science

General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.



Note

A characterisation of the least-fixed-point operator by dinaturality

Alex K. Simpson*

Department of Computer Science, University of Edinburgh, JCMB, The King's Buildings, Edinburgh EH9 3JZ, UK

Communicated by P. Freyd

Received August 1992

Revised January 1993

Abstract

Simpson, A.K., A characterisation of the least-fixed-point operator by dinaturality, Theoretical Computer Science 118 (1993) 301–314.

The paper addresses the question of when the least-fixed-point operator, in a cartesian-closed category of domains, is characterised as the unique dinatural transformation from the exponentiation bifunctor to the identity functor. We give a sufficient condition on a cartesian-closed full subcategory of the category of algebraic cpos for the characterisation to hold. The condition is quite mild, and the least-fixed-point operator is so characterised in many of the most commonly used categories of domains. By using retractions, the characterisation extends to the associated cartesian-closed categories of continuous cpos. However, dinaturality does not always characterise the least-fixed-point operator. We show that in cartesian-closed full subcategories of the category of continuous lattices the characterisation fails.

1. Introduction

Mulry [8] has shown that, under general conditions on a category of domains, the least-fixed-point operator, $\text{lfp}_D: D^D \rightarrow D$, is a dinatural transformation from the

Correspondence to: A.K. Simpson, Department of Computer Science, University of Edinburgh, JCMB, The King's Buildings, Edinburgh EH9 3JZ, UK. Email: als@dcs.edinburgh.ac.uk.

*Supported by SERC grant no. 90311820.

exponentiation bifunctor to the identity functor, i.e. for any $f:D \rightarrow E$, $g:E \rightarrow D$, $\text{lfp}_E(f \circ g) = f(\text{lfp}_D(g \circ f))$. He then asks whether, in any of the usual categories of domains, the property of dinaturality characterises the least-fixed-point operator. The question is of interest as, not only does the property of dinaturality arise from purely categorical considerations, but also such a characterisation would determine equationally the inequationally defined least-fixed-point operator. In this paper we answer Mulry's question for many of the most commonly used categories of domains.

Following a well-established tradition (see [3] for motivation), we consider cartesian-closed full subcategories of the category of algebraic cpos (algebraic directed-complete partial orders with least element) and continuous functions. If such a category has certain pushouts then we can answer Mulry's question in the affirmative: the least-fixed-point operator is indeed the unique dinatural transformation of appropriate type. The condition is rather weak and the required pushouts exist in, for example, the category of algebraic bounded-complete cpos [3, Chapter 5], the category of bifinite cpos [3, Chapter 10], and the category of algebraic L-domains [5]. Furthermore, using retracts, the characterisations extend to the associated categories of continuous cpos.

However, in the category of algebraic lattices (and continuous functions) the relevant pushouts do not exist. This is no accident. As we shall see, the least-fixed-point operator is not the unique dinatural transformation (between the appropriate bifunctors) in this category. In fact we give a more general counterexample that works for any (nontrivial) cartesian-closed full subcategory of the category of continuous lattices.

Section 2 presents the well-known connections between least-fixed-point operators and dinatural transformations in an abstract setting. In Section 3 we move to the concrete, giving the basic definitions and results we require from domain theory. Section 4 contains the main results, two theorems giving conditions under which Mulry's question can be answered in the affirmative. In Section 5 we consider many familiar categories of domains, applying the results of Section 4 to all but the various categories of lattices for which, in contrast, we answer Mulry's question in the negative.

2. Least-fixed-point operators and fix-dinaturals

The connections between (least-)fixed-point operators and dinatural transformations are best introduced in an abstract setting. Let \mathcal{C} be a **Poset**-enriched category which is cartesian-closed (in the **Poset**-enriched sense) such that $\mathcal{C}(\mathbf{1}, -): \mathcal{C} \rightarrow \mathbf{Poset}$ is faithful.

Recall that a **Poset**-enriched category is a category each hom-set of which is partially ordered such that composition is monotone in each argument. The requirement that $\mathcal{C}(\mathbf{1}, -)$ be faithful (well-pointedness) enables us to regard \mathcal{C} as a concrete category of partially ordered sets (of global elements) and monotone functions. We

will exploit this possibility, using set-theoretic notation for global elements and function application. The enriched aspect of cartesian closure means that the elementwise order on E^D is inherited from the order on the hom-set $\mathcal{C}(D, E)$. Note that we do not require that $\mathcal{C}(\mathbf{1}, -)$ reflect the partial order (equivalently that exponentials have the pointwise partial order), although this does hold in all the examples considered below.

A *fixed-point operator* Y on \mathcal{C} is an $\text{ob}(\mathcal{C})$ -indexed family of morphisms $D^D \xrightarrow{Y_D} D$ such that, for all $D \xrightarrow{f} D$, $Y_D(f) = f(Y_D(f))$. Y is the *least-fixed-point operator* if it is a fixed-point operator and $x = f(x) \in D$ implies $Y_D(f) \leq x$. By well-pointedness, the least-fixed-point operator, if it exists, is unique.

In any cartesian-closed category the exponentiation operation, $(-)^{(-)}$, is bifunctorial (it gives a functor from $\mathcal{C}^{\text{op}} \times \mathcal{C}$ to \mathcal{C}). A *dinatural transformation*, Y , from $(-)^{(-)}$ to the identity functor is an $\text{ob}(\mathcal{C})$ -indexed family of morphisms $D^D \xrightarrow{Y_D} D$ such that the diagram below commutes.

$$\begin{array}{ccc}
 & D^D & \xrightarrow{Y_D} D \\
 (-)^f \nearrow & & \downarrow f \\
 D^E & & \\
 f^{(-)} \searrow & & \\
 & E^E & \xrightarrow{Y_E} E
 \end{array}$$

This is an instance of the general definition of dinatural transformation between arbitrary bifunctors [7]. However, we will only be interested in dinatural transformations from $(-)^{(-)}$ to the identity functor, so henceforth we omit the functorial information referring to such dinaturals as *fix-dinaturals*. By well-pointedness, fix-dinaturality is equivalent to:

$$\text{for all } D \xrightarrow{f} E, \quad \text{for all } E \xrightarrow{g} D, \quad Y_E(f \circ g) = f(Y_D(g \circ f)).$$

The propositions below are well known (see e.g. [1, 8]). The first justifies the term “fix-dinatural”.

Proposition 2.1. *Any fix-dinatural transformation is a fixed-point operator.*

Proof. Let Y be a fix-dinatural transformation. Then it is immediate from the definition of dinaturality that $Y_D(f) = Y_D(f \circ 1_D) = f(Y_D(1_D \circ f)) = f(Y_D(f))$. \square

Lemma 2.2. *Let Y be a fixed-point operator. If, for all $D \xrightarrow{f} E$ and $E \xrightarrow{g} D$, $Y_E(f \circ g) \leq f(Y_D(g \circ f))$ then Y is a fix-dinatural.*

Proof. Take any $D \xrightarrow{f} E$ and $E \xrightarrow{g} D$. We need only show that $f(Y_D(g \circ f)) \leq Y_E(f \circ g)$. But $Y_D(g \circ f) \leq g(Y_E(f \circ g))$. So $f(Y_D(g \circ f)) \leq f(g(Y_E(f \circ g))) = Y_E(f \circ g)$, as $Y_E(f \circ g)$ is a fixed point of $f \circ g$. \square

Corollary 2.3. *The least-fixed-point operator, if it exists, is a fix-dinatural transformation.*

Proof. Let Y be the least-fixed-point operator. Take any $D \xrightarrow{f} E$ and $E \xrightarrow{g} D$. Then $f(Y_D(g \circ f)) = f(g(f(Y_D(g \circ f))))$. But this shows that $f(Y_D(g \circ f))$ is a fixed-point of $f \circ g$. So, as Y is the least-fixed-point operator, $Y_E(f \circ g) \leq f(Y_D(g \circ f))$. \square

It is not, in general, the case that the least-fixed-point operator is characterised as the unique fix-dinatural. In the category of complete lattices and monotone functions there are (at least) two fix-dinaturals: the least-fixed-point operator and the greatest-fixed-point operator.

3. Domain-theoretic preliminaries

Let D be a partially ordered set. A subset $X \subseteq D$ is *directed* if it is nonempty and every pair of elements in X has an upper bound in X . D is a *directed-complete partial order (cpo)* if it has a least element (which we denote \perp_D) and every directed $X \subseteq D$ has a least upper bound (lub), $\bigvee X \in D$. A function $f: D \rightarrow E$ between two cpos, is *continuous* if, for all directed $X \subseteq D$, $f(\bigvee X) = \bigvee f(X)$. Any continuous function is monotone. We write $[D \rightarrow E]$ for the set of all continuous functions from D to E . **Cpo** is the category of all cpos with continuous functions for morphisms. It is a well-pointed, **Poset**-enriched cartesian-closed category with the exponential E^D given by $[D \rightarrow E]$ with the point-wise ordering. As is well known, **Cpo** has a least-fixed-point operator lfp , where $\text{lfp}_D: [D \rightarrow D] \rightarrow D$ is defined by

$$\text{lfp}_D(f) = \bigvee \{f^n(\perp_D) \mid n \geq 0\}.$$

By Corollary 2.3, lfp is a fix-dinatural in **Cpo**.

The *way-below* relation, \ll , on a cpo D is defined by:

$$d \ll e \quad \text{iff for all directed } X \subseteq D, \text{ if } e \leq \bigvee X \text{ then there exists } x \in X \text{ such that } d \leq x.$$

The way-below relation has the expected properties: $d \ll e$ implies $d \leq e$; and $d' \leq d \ll e \leq e'$ implies $d' \ll e'$. D is *continuous* if, for all $d \in D$, the set $\{d' \in D \mid d' \ll d\}$ is directed with lub d . A continuous cpo D is *countably based* if there exists a countable $B \subseteq D$ such that, for all $d \in D$, $B \cap \{d' \in D \mid d' \ll d\}$ is directed with lub d . In a continuous cpo the way-below relation is dense: if $d \ll e$ then there exists x such that $d \ll x \ll e$ [4, Proposition 1.8]. A function $f: D \rightarrow E$ from a continuous cpo D is continuous if and only if, for all $d \in D$, $f(d) = \bigvee_{d' \ll d} f(d')$ [4, Proposition 1.12].

An element $d \in D$ is *compact* if $d \leq d$. We write $K(D)$ for the set of compact elements of D . D is *algebraic* if, for all $d \in D$, the set $K(D) \cap \downarrow d$ (where $\downarrow d = \{e \in D \mid e \leq d\}$) is directed with lub d . Any algebraic cpo is continuous. An algebraic cpo is countably based if and only if its set of compact elements is countable. We will require one nontrivial fact concerning algebraic cpos (due to Jung).

Proposition 3.1 (Jung [4, Proposition 1.41]). *If $[D \rightarrow D]$ is algebraic then, for all compact $f \in [D \rightarrow D]$ and arbitrary $d \in D$, $f(d)$ is compact in D .*

Define **AlgCpo** and **ContCpo** to be the full categories of **Cpo** with, respectively, algebraic cpos and continuous cpos for objects. Neither of these categories is cartesian-closed. However, their cartesian-closed full subcategories have been extensively studied, and the largest ones identified [9, 5, 6].

Let D and E be cpos. D is a *retract* of E if there exist continuous $f: D \rightarrow E$ and $g: E \rightarrow D$ such that $g \circ f = 1_D$. If, in addition, $f \circ g \leq 1_E$ then f is called an *embedding* and g its associated *projection* (each of f and g being determined by the other). If \mathcal{K} is a full subcategory of **Cpo** then define the category $\mathbf{R}\mathcal{K}$ to be the full subcategory of **Cpo** whose objects are all cpos that are retracts of objects of \mathcal{K} .

Proposition 3.2. *If \mathcal{K} is a cartesian-closed full subcategory of **Cpo** then:*

- (1) *The cartesian-closed structure of \mathcal{K} is inherited from **Cpo**.*
- (2) *\mathcal{K} has a least-fixed-point operator, lfp , which is a fix-dinatural.*
- (3) *$\mathbf{R}\mathcal{K}$ is cartesian-closed.*
- (4) *If \mathcal{K} is a full subcategory of **AlgCpo** then $\mathbf{R}\mathcal{K}$ is a full subcategory of **ContCpo**.*

Proof. Statement (1) is (essentially) Lemma 5 of [9]. For (2), we know (from (1)) that, for any object D of \mathcal{K} , D^D is (isomorphic to) $[D \rightarrow D]$, so the components of the least-fixed-point operator are given by $\text{lfp}_D: [D \rightarrow D] \rightarrow D$. It also follows from 1 that \mathcal{K} is a well-pointed, **Poset**-enriched cartesian-closed category, so the fix-dinaturality of lfp follows from Corollary 2.3 above. Statement (3) is (essentially) Theorem 1.23 of [4]. Statement (4) is an easy corollary of Proposition 1.16 of [4]. \square

4. Results

This section contains the main results of the paper. Theorem 4.7 gives a sufficient condition under which the least-fixed-point operator in a cartesian-closed full subcategory of **AlgCpo** is characterised as the unique fix-dinatural in that category. The easy Theorem 4.8 enables the characterisation to be extended to suitable cartesian-closed full subcategories of **ContCpo**.

First we introduce the main construction involved in the characterisation proof. A *strict finite chain* in D is a finite subset $C = \{c_0, c_1, \dots, c_n\} \subseteq D$ such that $\perp_D = c_0 < c_1 < \dots < c_n$ (where $n \geq 0$). The *cokernel* (in **Cpo**) of the strict finite chain $C \subseteq D$

is the following cpo:

$$E = \{\langle 0, c \rangle \mid c \in C\} \cup \{\langle 1, d \rangle \mid d \in D \setminus C\} \cup \{\langle -1, d \rangle \mid d \in D \setminus C\},$$

ordered by:

$$\langle \sigma, d \rangle \leq \langle \tau, e \rangle \quad \text{iff } d \leq e \text{ and either } \sigma = \tau \text{ or there exists } c \in C \text{ such that } d \leq c \leq e.$$

It is straightforward to check that \leq is a partial order. Directed completeness will be established below.

Let $C = \{c_0, c_1, \dots, c_n\} \subseteq D$ and E be as above. We will characterise directed sets in E and their lubs. Yet Y be an arbitrary subset of E . A subset $Y' \subseteq Y$ is *cofinal* if, for all $y \in Y$, there exists $y' \in Y'$ such that $y \leq y'$. If $Y' \subseteq Y$ is cofinal then clearly the upper bounds (in E) of Y and Y' coincide. Also Y' is directed if and only if Y is. Consider the functions $g: D \rightarrow E$ and $-g: D \rightarrow E$ defined by:

$$g(d) = \begin{cases} \langle 0, d \rangle & \text{if } d \in C, \\ \langle 1, d \rangle & \text{if } d \notin C, \end{cases} \quad -g(d) = \begin{cases} \langle 0, d \rangle & \text{if } d \in C, \\ \langle -1, d \rangle & \text{if } d \notin C. \end{cases}$$

Clearly g and $-g$ are both monotone and reflect the order (i.e. $g(d) \leq g(e)$ implies $d \leq e$). Also $g(d) = \langle \sigma, d \rangle \in E$ if and only if $\sigma \neq -1$; similarly $-g(d) = \langle \sigma, d \rangle \in E$ if and only if $\sigma \neq 1$.

Proposition 4.1. *A subset $Y \subseteq E$ is directed if and only if there exists a directed $X \subseteq D$ such that either $g(X)$ is a cofinal subset of Y or $-g(X)$ is.*

Proof. Suppose that $Y \subseteq E$ is directed. Suppose, for contradiction, that there is no $X \subseteq D$ such that $g(X)$ or $-g(X)$ is a cofinal subset of Y . Then it is easy to see that there must be two elements $\langle 1, d \rangle$ and $\langle -1, e \rangle$ in Y which do not have an upper bound in Y , contradicting the directedness of Y . So indeed, for some $X \subseteq D$, $g(X)$ or $-g(X)$ is a cofinal subset of Y . That X is directed follows from the directedness of $g(X)$ or $-g(X)$ as appropriate.

Conversely, suppose there exists a directed $X \subseteq D$ such that, without loss of generality, $g(X) \subseteq Y$ is cofinal. Clearly Y is nonempty. Suppose that $\langle \sigma, d \rangle$ and $\langle \tau, e \rangle$ are in Y . We must show they have an upper bound in Y . Let $x \in X$ be such that $\langle \sigma, d \rangle \leq g(x)$ (x exists as $g(X)$ is cofinal). Similarly, let $y \in X$ be such that $\langle \tau, e \rangle \leq g(y)$. Now X is directed so x and y have some upper bound $z \in X$. Clearly, $g(z)$ is an upper bound in Y of $\{g(x), g(y)\}$ and hence of $\{\langle \sigma, d \rangle, \langle \tau, e \rangle\}$. \square

Lemma 4.2. *If $X \subseteq D$ and $\langle -1, d \rangle \in E$ is an upper bound of $g(X)$ then there exists $c \in C$ such that c is an upper bound of X and $c \leq d$.*

Proof. Suppose that $X \subseteq D$ and $\langle -1, d \rangle \in E$ is an upper bound of $g(X)$. Then, for each $x \in X$, $g(x) \leq \langle -1, d \rangle$. Now $g(x) = \langle \tau, x \rangle$, where $\tau \neq -1$, so (by definition of order on E)

there exists $c_x \in C$ such that $x \leq c_x \leq d$. Define $c = \bigvee_{x \in X} c_x$. Then clearly $c \in C$ (as C is a finite chain), c is an upper bound of X and $c \leq d$. \square

Proposition 4.3. *Suppose $Y \subseteq E$ and, for some $X \subseteq D$, $g(X)$ (resp. $-g(X)$) is a cofinal subset of Y . Then Y has a lub in E if and only if X has a lub in D , in which case $\bigvee Y = g(\bigvee X)$ (resp. $-g(\bigvee X)$).*

Proof. Suppose, without loss of generality, that $g(X) \subseteq Y$ is cofinal.

If $\bigvee X$ exists then, by the monotonicity of g , $g(\bigvee X)$ is an upper bound of $g(X)$ and hence of Y . To see it is the lub, suppose $\langle \sigma, d \rangle$ is another upper bound of Y and hence of $g(X)$. Now d is an upper bound of X , so $\bigvee X \leq d$. Thus if $\sigma \neq -1$ then $g(\bigvee X) \leq g(d) = \langle \sigma, d \rangle$ as required. We must still show that $g(\bigvee X) \leq \langle \sigma, d \rangle$ if $\sigma = -1$. But then, by Lemma 4.2, there exists $c \in C$ such that c is an upper bound of X and $c \leq d$. And so $g(\bigvee X) \leq g(c) = \langle 0, c \rangle \leq \langle -1, d \rangle$ as required.

Conversely, suppose that $\bigvee Y$ exists. Setting $\langle \sigma, d \rangle = \bigvee Y = \bigvee g(X)$, we must show that $d = \bigvee X$ and $\sigma \neq -1$. First suppose that e is an upper bound of X . Then, by monotonicity, $g(e)$ is an upper bound of $g(X)$. So $\langle \sigma, d \rangle \leq g(e)$ and hence $d \leq e$. Thus, indeed, $d = \bigvee X$. Now suppose that $\sigma = -1$. Then, by Lemma 4.2, there exists $c \in C$ such that c is an upper bound of X and $c \leq d$. So, as $d = \bigvee X$, $c = d$. But then, as $c \in C$, $\sigma = 0$, a contradiction. So indeed $\sigma \neq -1$. \square

It is immediate from Propositions 4.1 and 4.3 that E is a cpo and that both g and $-g$ are continuous.

We now justify our choice of terminology for E . Let $i: C \rightarrow D$ be the inclusion of C in D . Recall the cokernel pair of i is the pushout of i along itself [7, p. 66].

Proposition 4.4. *The cokernel pair of i in \mathbf{Cpo} is $g, -g: D \rightarrow E$.*

Proof. First it is obvious that $g \circ i = -g \circ i$. For the universal property, let E' be any cpo, and $g', -g': D \rightarrow E'$ be continuous functions such that $g' \circ i = -g' \circ i$, i.e. for all $c \in C$, $g'(c) = -g'(c)$. We must show that there is a unique continuous $m: E \rightarrow E'$ such that $g' = m \circ g$ and $-g' = m \circ -g$. Clearly the unique function satisfying the equalities is

$$m(\langle \sigma, d \rangle) = \begin{cases} g'(d) & \text{if } \sigma \neq -1, \\ -g'(d) & \text{if } \sigma \neq 1. \end{cases}$$

This is a good definition as the only conflict is when $\sigma = 0$ in which case $d \in C$ and so $g'(d) = -g'(d)$. It remains to show that m is continuous. Let $Y \subseteq E$ be directed. If, for some directed $X \subseteq D$, $g(X) \subseteq Y$ is cofinal then $m(\bigvee Y) = m(g(\bigvee X)) = g'(\bigvee X) = \bigvee g'(X) = \bigvee m(g(X)) = \bigvee m(Y)$ (using Proposition 4.3 and the continuity of g'). A similar argument establishes that $m(\bigvee Y) = \bigvee m(Y)$ if $-g(X) \subseteq Y$ is cofinal. By Proposition 4.1, this covers all cases. \square

Proposition 4.5. *Suppose D is algebraic. Then E is algebraic if and only if every element of C is compact in D , in which case $K(E) = \{ \langle \sigma, d \rangle \in E \mid d \in K(D) \}$.*

Proof. Suppose that every element in C is compact. We first show that $K(E) = \{\langle \sigma, d \rangle \in E \mid d \in K(D)\}$.

Suppose that $\langle \sigma, d \rangle$ is compact in E and $X \subseteq D$ is a directed set such that $d \leq \bigvee X$. If $\sigma \neq -1$ then $g(X) \subseteq E$ is directed such that $\bigvee g(X) = g(\bigvee X) \geq g(d) = \langle \sigma, d \rangle$. Therefore, by the compactness of $\langle \sigma, d \rangle$, there exists $\langle \tau, x \rangle \in g(X)$ such that $\langle \sigma, d \rangle \leq \langle \tau, x \rangle$. But then we have found the required $x \in X$ such that $d \leq x$. A similar argument (using $-g$ in place of g) works for the case that $\sigma = -1$. So, either way, $d \in K(D)$.

For the converse inclusion, suppose d is compact in D . We show that $g(d)$ and $-g(d)$ are both compact in E . For $g(d)$ suppose $Y \subseteq E$ is directed with $g(d) \leq \bigvee Y$. Now if $g(X) \subseteq Y$ is cofinal for some directed $X \subseteq D$ then $g(d) \leq \bigvee Y = g(\bigvee X)$ (by Proposition 4.3), so $d \leq \bigvee X$. But then, by the compactness of d , there exists $x \in X$ such that $d \leq x$. So $g(x) \in Y$ is the required element such that $g(d) \leq g(x)$. Alternatively, if $-g(\bigvee X) \subseteq Y$ is cofinal then $g(d) \leq \bigvee Y = -g(\bigvee X)$. By the definition of order on E , there exists $c \in C$ such that $d \leq c \leq \bigvee X$. But then, as c is compact, there exists $x \in X$ such that $c \leq x$. So $g(d) \leq -g(x) \in Y$. By Proposition 4.1, we have covered all possibilities for Y . So $g(d)$ is indeed compact. The compactness of $-g(d)$ is by a similar argument.

We now show that E is algebraic. Let $\langle \sigma, d \rangle$ be an arbitrary element of E . We must show that $Y = K(E) \cap \downarrow \langle \sigma, d \rangle$ is directed with lub $\langle \sigma, d \rangle$. For compact $\langle \sigma, d \rangle$ this is trivial. Suppose then that $\langle \sigma, d \rangle$ is not compact. Thus d is not compact in D , hence $d \notin C$ and so $\sigma \neq 0$. Define $X = K(D) \cap \downarrow d$, which, by the algebraicity of D , is directed with lub d . If $\sigma = 1$ then it is easy to see that $g(X) \subseteq Y$ and is cofinal (similarly for $-g(X)$ if $\sigma = -1$). So, by Propositions 4.1 and 4.3, Y is directed with lub $\langle 1, d \rangle$, as required.

It remains to show that if, for some $c \in C$, c is not compact then E is not algebraic. Suppose c_k ($1 \leq k \leq n$) is such an element. Consider $X = \{d \in K(D) \mid c_{k-1} < d < c_k\}$, which, by the algebraicity of D , is directed with lub c_k . Then, by Propositions 4.1 and 4.3, $g(X)$ and $-g(X)$ are both directed with lub $\langle 0, c_k \rangle$. It is now easy to see that $\langle 0, c_{k-1} \rangle$ is the highest possible compact element in E below $\langle 0, c_k \rangle$. For, if $c_{k-1} < d < c_k$, then $\langle 1, d \rangle \leq \bigvee (-g(X))$, but there is clearly no $\langle \sigma, x \rangle \in -g(X)$ with $\langle 1, d \rangle \leq \langle \sigma, x \rangle$, as $\sigma = -1$. Similarly, $g(X)$ shows that $\langle -1, d \rangle$ is not compact. So $\langle 0, c_k \rangle$ is not the lub of the compact elements lower than it. \square

Proposition 4.6. *If every element of C is compact in D then $g: D \rightarrow E$ is an embedding.*

Proof. The associated projection $h: E \rightarrow D$ is defined by

$$h(\langle \sigma, d \rangle) = \begin{cases} d & \text{if } \sigma \neq -1, \\ \bigvee \{c \in C \mid c \leq d\} & \text{if } \sigma = -1. \end{cases}$$

Clearly $h \circ g = 1_D$ and $g \circ h \leq 1_E$. The continuity of h follows straightforwardly from the continuity of $d \mapsto \bigvee \{c \in C \mid c \leq d\}: D \rightarrow D$, which we now establish. Clearly, this is monotone, so it is sufficient to show that, for any directed $X \subseteq D$, $\bigvee \{c \in C \mid c \leq \bigvee X\} \leq \bigvee_{x \in X} \bigvee \{c \in C \mid c \leq x\}$. But, as C is a finite chain, $\bigvee \{c \in C \mid c \leq \bigvee X\} = c'$ for

some $c' \in C$ such that $c' \leq \bigvee X$. Then, by the compactness of c' , there exists $x \in X$ such that $c' \leq x$. Clearly, $c' \leq \bigvee_{x \in X} \bigvee \{c \in C \mid c \leq x\}$, as required. \square

We now give the main theorem of the paper. Let \mathcal{K} be a cartesian-closed full subcategory of **AlgCpo**. We say that \mathcal{K} is *closed under cokernels (in Cpo) of strict finite chains of compact elements* if, for every algebraic cpo D in \mathcal{K} , for every strict finite chain $C \subseteq D$ of compact elements, the cokernel of $C \subseteq D$ (as defined above) is an object of \mathcal{K} . By Proposition 4.5, we cannot hope for \mathcal{K} to be closed under cokernels of chains of noncompact elements.

Theorem 4.7. *If \mathcal{K} is a cartesian-closed full subcategory of **AlgCpo** and \mathcal{K} is closed under cokernels of strict finite chains of compact elements, then the least-fixed-point operator is the unique fix-dinatural in \mathcal{K} .*

Proof. By Proposition 3.2, \mathcal{K} has a least-fixed-point operator, lfp , which is fix-dinatural. For the converse, let Y be any fix-dinatural in \mathcal{K} and let D be any object of \mathcal{K} . We must show that $Y_D = \text{lfp}_D$. However, a continuous function whose domain is an algebraic cpo is determined by its behaviour on compact elements. So we need only show that, for compact f in $[D \rightarrow D]$, $Y_D(f) = \text{lfp}_D(f)$.

Accordingly, let $f \in [D \rightarrow D]$ be compact. Now $f(\text{lfp}_D(f)) = \text{lfp}_D(f)$, so, by Proposition 3.1, $\text{lfp}_D(f)$ is compact. Also $\text{lfp}_D(f) = \bigvee \{\perp, f(\perp), f^2(\perp), \dots\}$, so, by its compactness, there exists (a least) $n \geq 0$ such that $\text{lfp}_D(f) = f^n(\perp)$. So $C = \{\perp, f(\perp), \dots, f^n(\perp)\}$ is a strict finite chain and, by Proposition 3.1, all its elements are compact.

Let E be the cokernel of $C \subseteq D$, and let $g: D \rightarrow E$ be as above. We now define the, as it were, symmetric extension of f to E . This is the endofunction $\bar{f}: E \rightarrow E$ given by

$$\bar{f}(\langle \sigma, d \rangle) = \begin{cases} \langle 0, f(d) \rangle & \text{if } f(d) \in C, \\ \langle \sigma, f(d) \rangle & \text{if } f(d) \notin C. \end{cases}$$

\bar{f} is well defined as $c \in C$ implies $f(c) \in C$. It is also clearly continuous (in fact it can be obtained from the universal property of E). Further, it is clear that

$$g \circ f = \bar{f} \circ g. \tag{i}$$

The symmetry of \bar{f} enables us to determine the value of $Y_E(\bar{f})$. Consider the function $-1: E \rightarrow E$ defined by

$$-1(\langle \sigma, d \rangle) = \langle -\sigma, d \rangle.$$

This is clearly continuous (again it can be obtained from the universal property of E). Clearly, $-1 \circ -1 = 1_E$ and $\bar{f} = -1 \circ \bar{f} \circ -1$, the latter equation being the formal statement of the symmetry of \bar{f} . So

$$\begin{aligned} Y_E(\bar{f}) &= Y_E(-1 \circ -1 \circ \bar{f}) \\ &= -1(Y_E(-1 \circ \bar{f} \circ -1)) \quad (\text{by dinaturality}) \\ &= -1(Y_E(\bar{f})). \end{aligned}$$

Thus $Y_E(\bar{f})$ is of the form $\langle 0, c \rangle$ for some $c \in C$. But, by Proposition 2.1, $Y_E(\bar{f})$ is a fixed point of \bar{f} and so can only be $\langle 0, f^n(\perp) \rangle$. Thus we have

$$Y_E(\bar{f}) = \langle 0, \text{lfp}_D(f) \rangle. \quad (\text{ii})$$

Now let $h: E \rightarrow D$ be the projection associated with g given by Proposition 4.6. Then

$$\begin{aligned} Y_D(f) &= h(g(Y_D(f \circ h \circ g))) && (\text{as } h \circ g = 1_D) \\ &= h(Y_E(g \circ f \circ h)) && (\text{by dinaturality}) \\ &= h(Y_E(\bar{f} \circ g \circ h)) && (\text{by (i) above}) \\ &\leq h(Y_E(\bar{f})) && (\text{as } g \circ h \leq 1_E) \\ &= h(\langle 0, \text{lfp}_D(f) \rangle) && (\text{by (ii) above}) \\ &= \text{lfp}_D(f). \end{aligned}$$

Therefore, $Y_D(f) \leq \text{lfp}_D(f)$. But, by Proposition 2.1, $Y_D(f)$ is a fixed point of f , so $Y_D(f) = \text{lfp}_D(f)$ as required. \square

Unfortunately, due to the role of compact elements in the above proof, we do not know how to extend the techniques to deal directly with categories of non-algebraic cpos. However, in view of Proposition 3.2, there is an indirect way of extending the characterisation to cartesian-closed full subcategories of **ContCpo**.

Theorem 4.8. *Let \mathcal{K} be a cartesian-closed full subcategory of **Cpo**. If the least-fixed-point operator is the unique fix-dinatural in \mathcal{K} then the least-fixed-point operator is also the unique fix-dinatural in $\mathbf{R}\mathcal{K}$.*

Proof. Let Y be a fix-dinatural in $\mathbf{R}\mathcal{K}$. It is easy to see that the restriction of Y to objects of \mathcal{K} is a fix-dinatural in \mathcal{K} . So, for any cpo E in \mathcal{K} , $Y_E = \text{lfp}_E$. Now let D be an arbitrary cpo in $\mathbf{R}\mathcal{K}$. Then there exists E in \mathcal{K} and continuous functions $g: D \rightarrow E$ and $h: E \rightarrow D$ such that $h \circ g = 1_D$. Consider any continuous $f: D \rightarrow D$. Then

$$\begin{aligned} Y_D(f) &= Y_D(h \circ g \circ f) && (\text{as } h \circ g = 1_D) \\ &= h(Y_E(g \circ f \circ h)) && (\text{by dinaturality of } Y) \\ &= h(\text{lfp}_E(g \circ f \circ h)) && (\text{as } Y_E = \text{lfp}_E) \\ &= \text{lfp}_D(h \circ g \circ f) && (\text{by dinaturality of lfp}) \\ &= \text{lfp}_D(f). \quad \square \end{aligned}$$

5. Examples

In this section we consider many cartesian-closed full subcategories of **AlgCpo** and **ContCpo**, determining whether or not the least-fixed-point operator is characterised

as the unique fix-dinatural. Theorem 5.1 shows that the most commonly used cartesian-closed full subcategories of **AlgCpo** do satisfy the condition of Theorem 4.7. We thereby obtain a good collection of categories (of both algebraic and continuous cpos) in which the least-fixed-point operator is indeed the unique fix-dinatural. However, categories of algebraic and continuous lattices are not amenable to the techniques so far developed. In Theorem 5.3 we show that there is a second fix-dinatural in these categories, so it is no accident that the results of the last section are not applicable.

We briefly review the definitions (by just specifying the objects) of some of the most important cartesian-closed full subcategories of **AlgCpo** and **ContCpo**. Let D be a cpo. D is *bounded-complete* if each subset $X \subseteq D$ with an upper bound in D has a lub in D . **AlgBC** and ω **AlgBC**, the categories of algebraic bounded-complete cpos and countably based algebraic bounded-complete cpos, are both cartesian-closed [3, Chapter 5]. The retracts of (countably based) algebraic bounded-complete cpos are just the (countably based) continuous bounded-complete cpos and so the categories **ContBC** and ω **ContBC**, of continuous bounded-complete cpos and countably based continuous bounded-complete cpos, are cartesian-closed (by Proposition 3.2). D is *bifinite* if there is a directed (under the pointwise ordering) set of functions $\{f_i \in [D \rightarrow D]\}_{i \in I}$ such that each f_i is idempotent with finite image and $\bigvee_{i \in I} \{f_i\} = 1_D$. It is a fact that any bifinite cpo is algebraic. **Bifin** and ω **Bifin**, the categories of bifinite cpos and countably based bifinite cpos, are both cartesian-closed [3, Chapter 10]. D is an *L-domain* if, for every $x \in D$, the set $\downarrow x$ is a complete lattice under the induced ordering. **AlgL**, the category of algebraic L-domains, is cartesian-closed [5]. (The category of countably based algebraic L-domains is not cartesian-closed.) The retracts of algebraic L-domains are just the continuous L-domains [4, Proposition 4.20], and so the category **ContL**, of continuous L-domains, is cartesian-closed.

Theorem 5.1. *The following cartesian-closed full subcategories of **AlgCpo** are closed under cokernels of strict finite chains of compact elements: **AlgBC**, ω **AlgBC**, **Bifin**, ω **Bifin** and **AlgL**.*

Proof. Let D be an algebraic cpo, $C \subseteq D$ a strict finite chain of compact elements and let E be the cokernel of $C \subseteq D$, constructed as above. By Proposition 4.5, we know that E is algebraic and is countably based if D is.

For **AlgL** we need only show that if D is an L-domain then so is E . Let $\langle \sigma, d \rangle$ be an arbitrary element of E and let Y be any subset of $\downarrow \langle \sigma, d \rangle$. We must show that Y has a lub in $\downarrow \langle \sigma, d \rangle$. If, for some $X \subseteq D$, $g(X) \subseteq Y$ is cofinal then clearly $X \subseteq \downarrow d$ has a lub (in $\downarrow d$), x say, and $g(x)$ is the lub of Y in $\downarrow \langle \sigma, d \rangle$. A similar argument deals with the case that $-g(X) \subseteq Y$ is cofinal. If there is no $X \subseteq D$ such that either $g(X) \subseteq Y$ or $-g(X) \subseteq Y$ is cofinal, then it is straightforward to check that the lub of Y in $\downarrow \langle \sigma, d \rangle$ is given by $\langle 0, \bigwedge \{c \in C \mid \langle 0, c \rangle \text{ is an upper bound of } Y\} \rangle$.

Suppose D is bifinite, as witnessed by $\{f_i\}_{i \in I}$. Since $\bigvee_{i \in I} \{f_i\} = 1_D$, it is straightforward to show that, for every compact $d \in D$, there is an $i \in I$ such that d is in the image of

f_i . Therefore, as $\{f_i\}_{i \in I}$ is directed, there is some i such that C is contained in the image of f_i . So $J = \{j \in I \mid C \text{ is contained in the image of } f_j\}$ is nonempty. Further, it is easy to see that $\{f_j\}_{j \in J}$ is directed with $\bigvee_{j \in J} \{f_j\} = 1_D$. Define, for each $j \in J$, $\tilde{f}_j: E \rightarrow E$ by

$$\tilde{f}_j(\langle \sigma, d \rangle) = \begin{cases} \langle 0, f_j(d) \rangle & \text{if } f_j(d) \in C, \\ \langle \sigma, f_j(d) \rangle & \text{if } f_j(d) \notin C. \end{cases}$$

These functions are well defined as $c \in C$ implies $f_j(c) = c$, as is clear from the idempotency of f_j . Moreover, the \tilde{f}_j are easily seen to be idempotent with finite image, and $\{\tilde{f}_j\}_{j \in J}$ is directed with $\bigvee_{j \in J} \{\tilde{f}_j\} = 1_E$. So E is bifinite. Thus **Bifin** and ω **Bifin** contain the required cokernels.

For **AlgBC** and ω **AlgBC** we need only show that if D is bounded-complete then so is E . This is straightforward and is left for the reader to verify. \square

Corollary 5.2. *The least-fixed-point operator is characterised as the unique fix dinatural in the following cartesian-closed full subcategories of **AlgCpo**: **AlgBC**, ω **AlgBC**, **Bifin**, ω **Bifin**, **AlgL**; and in the following cartesian-closed full subcategories of **ContCpo**: **ContBC**, ω **ContBC**, **RBifin**, **R ω Bifin**, **ContL**.*

Proof. Immediate from the above theorem together with Theorems 4.7 and 4.8. \square

The above examples show that Theorems 4.7 and 4.8 are widely applicable. However, the condition of Theorem 4.7 is not universally satisfied. The categories **AlgCL** and ω **AlgCL**, of algebraic (complete) lattices and countably based algebraic lattices, are two cartesian-closed full subcategories of **AlgCpo** that are not closed under cokernels of strict finite chains of compact elements (as is easily seen). In fact, as we now show, the least-fixed-point operator is not the unique fix-dinatural in these categories.

For greater generality, we work with the category **ContCL** of continuous lattices (which is a cartesian-closed full subcategory of **ContCpo**). A natural candidate for a second fix-dinatural would be an operator finding the greatest fixed-points of endomorphisms, but the function doing so is not continuous. However, there is a greatest continuous fixed-point operator, gcfp , given by the formula

$$\text{gcfp}_D(f) = \bigvee \{f^n(d) \mid d \ll f(d), n \geq 0\}$$

(inspired by Bracho's formula for fixed-point-operators in **AlgBC** [2]).

Theorem 5.3. *gcfp is a fix-dinatural in **ContCL**.*

Proof. We first show the continuity of gcfp_D . As $[D \rightarrow D]$ is a continuous cpo, it is enough to show that $\text{gcfp}_D(f) = \bigvee_{f' \ll f} \text{gcfp}_D(f')$. But

$$\text{gcfp}_D(f) = \bigvee \{f^n(d) \mid d \ll f(d)\} = \bigvee_{f' \ll f} \bigvee \{f^n(d) \mid d \ll f'(d)\}.$$

The second equality holds because $d \ll f(d)$ if and only if there exists $f' \ll f$ such that $d \ll f'(d)$ (for the nontrivial left-to-right implication suppose $d \ll f(d)$, then there is, as \ll is dense, an interpolant d' such that $d \ll d' \ll f(d) = \bigvee_{f' \ll f} f'(d)$, from which it is easily seen that $d \ll f'(d)$ for some $f' \ll f$). Then,

$$\begin{aligned} \text{gcfp}_D(f) &= \bigvee_{f' \ll f} \bigvee \{f^n(d) \mid d \ll f'(d)\} \\ &= \bigvee_{f' \ll f} \bigvee_{f'' \ll f'} \bigvee \{(f'')^n(d) \mid d \ll f'(d)\} \\ &\quad \text{(by the continuity of } [D \rightarrow D]) \\ &= \bigvee_{f' \ll f} \bigvee \{(f')^n(d) \mid d \ll f'(d)\} \\ &\quad \text{(by continuity considerations)} \\ &= \bigvee_{f' \ll f} \text{gcfp}_D(f'). \end{aligned}$$

For fix-dinaturality, let D and E be continuous lattices. Let $f: D \rightarrow E$ and $g: E \rightarrow D$ be continuous. Then

$$\begin{aligned} \text{gcfp}_E(f \circ g) &= f(\bigvee \{(g \circ f)^n(g(e)) \mid e \ll f(g(e))\}) \\ &= f(\bigvee_{g' \ll g} \bigvee \{(g \circ f)^n(g(e)) \mid e \ll f(g'(e))\}). \end{aligned}$$

The left-hand equality holds because $\{(g \circ f)^n(g(e)) \mid e \ll f(g(e))\}$ is directed; the right-hand because $e \ll f(g(e))$ if and only if there exists $g' \ll g$ such that $e \ll f(g'(e))$ (proof as above). So, by continuity considerations, we have

$$\text{gcfp}_E(f \circ g) = f(\bigvee_{g' \ll g} \bigvee \{(g \circ f)^n(g'(e)) \mid e \ll f(g'(e))\}).$$

Below we show that $g' \ll g$ implies $g'(e) \ll g(e)$. From this it is clear that if $g' \ll g$ and $e \ll f(g'(e))$ then $g'(e) \ll g(f(g'(e)))$. So

$$\text{gcfp}_E(f \circ g) \leq f(\bigvee \{(g \circ f)^n(d) \mid d \ll g(f(d))\}) = f(\text{gcfp}_D(g \circ f)).$$

Thus $\text{gcfp}_E(f \circ g) \leq f(\text{gcfp}_D(g \circ f))$ and the fix-dinaturality of gcfp follows from Lemma 2.2.

It remains to show that $g' \ll g$ implies $g'(e) \ll g(e)$. However, suppose $g(e) \leq \bigvee X$ for some directed $X \subseteq D$. Define, for each $x \in X$, $g_x: E \rightarrow D$ by

$$g_x(y) = \begin{cases} g(y) & \text{if } y \not\leq e, \\ g(y) \wedge x & \text{if } y \leq e. \end{cases}$$

It is easy to show that each g_x is continuous (using the well-known continuity of the binary meet operation on continuous lattices) and that $\bigvee_{x \in X} g_x = g$. But if $g' \ll g$ then, for some $x \in X$, $g' \leq g_x$. So $g'(e) \leq g_x(e) \leq x$. Thus, indeed, $g'(e) \ll g(e)$. \square

Corollary 5.4. *The least-fixed-point operator is not the unique fix-dinatural in any nontrivial cartesian-closed full subcategory of **ContCL**.*

Proof. Let \mathcal{K} be a cartesian-closed full subcategory of **ContCL**. It is easy to see (using Proposition 3.2) that the restriction of gcfp to objects of \mathcal{K} is a fix-dinatural in \mathcal{K} .

However, gcfp differs from lfp on all but the trivial cpo . \square

All the results in this paper have been for cartesian-closed full subcategories of **ContCpo**. We do not know whether any of them can be extended to cover categories of noncontinuous cpo s. There is also one prominent gap in our knowledge of continuous cpo s: it remains an open question whether the least-fixed-point operator is the unique fix-dinatural in Jung's category of FS-domains [6].

It is worth remarking that the techniques of this paper extend beyond the case of full subcategories of **Cpo**. For example, using cokernels of strict ordinal-indexed chains, it can be shown that the least-fixed-point operator is the unique fix-dinatural in the category of cpo s and all monotone maps. The proof is straightforward as no considerations of continuity or algebraicity are involved. The proof for the category of finite pointed posets (i.e. those with least element) and all monotone functions is even easier (only finite chains are required). We note that an easier proof of the characterisation for **Bifin** can be obtained by extrapolation from the category of finite pointed posets.

Acknowledgment

I would like to thank Marcelo Fiore, Barry Jay and, especially, Gordon Plotkin for their suggestions.

References

- [1] E.S. Bainbridge, P.J. Freyd, A. Scedrov and P.J. Scott, Functorial polymorphism, *Theoret. Comput. Sci.* **70** (1990) 35–64.
- [2] F. Bracho, Finitary retracts and fixed points, in: *Proc. Internat. Coll. on the Formalization of Programming Concepts*, in: Lecture Notes in Computer Science, Vol. 107 (Springer, Berlin, 2nd ed., 1981); also available as Investigacones No. 257, IIMAS, Universidad Nacional Autonoma de Mexico.
- [3] C.A. Gunter, *Semantics of Programming Languages*. (MIT Press, Cambridge, MA, 1992).
- [4] A. Jung, *Cartesian Closed Categories of Domains*, Vol. 66, *CWI Tract*. (Centrum voor Wiskunde en Informatica, Amsterdam, 1989).
- [5] A. Jung, Cartesian closed categories of algebraic cpo s, *Theoret. Comput. Sci.* **70** (1990) 233–250.
- [6] A. Jung, The classification of continuous domains, in: *Proc. 5th Ann. Symp. on Logic in Computer Science* (1990) 35–40.
- [7] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics (Springer, Berlin, 1971).
- [8] P.S. Mulry, Categorical fixed-point semantics, *Theoret. Comput. Sci.* **70** (1990) 85–97.
- [9] M.B. Smyth, The largest cartesian closed category of domains, *Theoret. Comput. Sci.* **27** (1983) 109–119.